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and there find it to be always the same as before proved in (40) and (41), except the initial or special differentiation, which was generalized between (34) and (35). This generalization was based on the fixed nature of the operation in connection with the actual developments copied in (34) and (35); which prove the general Theorem, already described.

RECENT RESULTS IN THE STUDY OF LINKAGES.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Colonel Peaucellier's discovery of an exact rectilinear motion produced by means of jointed rods is interesting not only for its own sake but as having been the starting point of a new and beautiful branch of Geometrical Study. An account of Peaucellier's invention and of some of the earlier results of the Study of Linkages was contributed by the writer to the Analyst for March, 1875; the present article relates to more recent results due principly to the English Mathematicians who have worked in this new field.

The term *linkage* is employed by Prof. Sylvester to denote a net-work of jointed bars, such that one bar being fixed, any point of another bar will describe a definite locus while the system changes its shape, or undergoes *deformation.* Thus a jointed quadrilateral is a linkage; if two bars jointed to gether have their free ends jointed one to each of two bars, the system becomes a six-bar linkage. It is evident that in this way we may form a linkage of any *even* number of bars, but that we cannot form a linkage of an odd number of bars.

The joints of a linkage constitute a system of points bound by an even number of conditions, each of which asserts the invariability of the distance of two points of the system. A point rigidly connected with one of the bars, whether on or off the straight line connecting the joints, may be regarded as determined by its distance from the two joints, that is by two conditions of the same form: but when a joint is assumed on a bar already included in the linkage, these conditions are not counted in estimating the order of the linkage, which depends not upon the number of joints but solely upon the number of bars which are moveable relatively to one another. The several joints on a single bar are usually taken in a straight line, but Sylvester remarks that "the true view of the theory of Linkages is to consider each bar as carrying with it an indefinitely extended plane."

If one bar of a linkage is fixed in a plane, the joints on the fixed bar become fixed pivots, and the fixed bar may be removed. The system, now consisting of an odd number of bars, is called a *linkwork.* In this point of view, the study of a linkwork becomes a question of the relative motion of two bars of a linkage.

But the formation of a linkwork for the purpose of describing a given curve is more frequently effected by the employment of a linkage as a *transforming cell.* When a linkage is thus employed one point called the *fulcrum* is fixed in a plane, and two other points are regarded, the one as *tracer,* and the other as *follower*; then when the tracer describes a given curve the follower describes a curve into which the given curve is said to be transformed by the given cell. Thus the Peaucellier cell (See ANALYST, March, 1875) transforms a curve into its inverse. The Pantagraph in its simplest form is a four-bar cell, in which the follower describes a curve similar to that described by the tracer.

If the tracer of a cell with a fixed fulcrum be constrained by a linkwork to describe a known curve, we shall have formed a linkwork in which the follower describes the transformed curve. The simplest constraining linkwork is a single bar causing the tracer to describe a circle, hence primary interest attaches to the transformations of the circle, the Peaucellier rectilinear motion, for instance, is a case of "inversion" of the circle.

The distances of the tracer and follower from the fulcrum of a cell are called the *rays.* In the forms which have been studied the rays are, for the most part, in one straight line. The angle which this line makes with a fixed line in the plane being denoted by θ, and the rays by r and ρ, if we have the relation between r and ρ characteristic of the cell, we can readily pass from the polar equation of the curve described by the tracer to that of the curve into which it is transformed.

The simplest relation between r and ρ is $r = m\rho$ as in the Pantagraph. No cell in which r and ρ are connected by the general linear relation seems to have been devised; but Messrs Hart and Kempe have given 8-bar linkages in which one point moves always in the line of one of the bars. Such a linkage, the point in question being taken as fulcrum, and the extremities of the bar as tracer and follower, would be a cell in which $\rho - r =$ a constant. The simplest relation of the second degree is $\rho r =$ a constant, in which r and ρ are inversely proportional, as in the Peaucellier cell, and in Hart's reciprocater. (see ANALYST, Vol. II, p. 44.)

A six-bar cell, in which $\rho^2 - r^2 =$ a constant, called by Sylvester the Quadratic Binomial Extractor, may be reduced to the four-bar cell represented in Fig. 1, in which $AB = BO = BQ = a$ and $AC = CO = CP = b$.

The line joining A and O is evidently perpendicular at once to PO and to OQ. We therefore have P, O, and Q always in a straight line, and denoting PO by ρ and OQ by r

$$r^2 - \rho^2 = 4(a^2 - b^2). \; . \; . \; . \; . \; (1)$$

This cell may be used to extract the square root of a binomial quadratic expression mechanically.

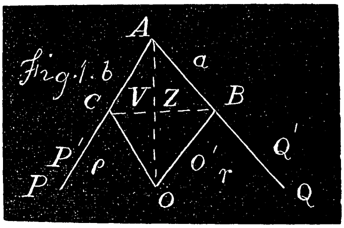


Fig. 1.

If we take $CP' = mb$, $BO' = BQ' = ma$, $P'O'$ and $O'Q'$ will always be in a straight line; and, denoting them by ρ and r, we shall have a more general quadratic cell. Denote the segments into which CB is divided by AO by v and z then $z^2 - v^2 = a^2 - b^2$. . . . . (2)

But $r = 2mz$ and $\rho = (1 + m)v + (1 - m)z$ hence

$$z = \frac{r}{2m} \quad \text{and} \quad v = \frac{\rho}{m+1} - \frac{(1-m)r}{2m(1+m)},$$

substituting in (2) and reducing

$$r^2 + (1 - m)r\rho - m\rho^2 = m(m + 1)^2(a^2 - b^2) = k^2. \; . \; . \; . \; . \; (3)$$

This is not however a general quadratic relation, but is subject to the condition that the coefficient of the product equals the sum of the coefficients of the squares. It will be observed that it is impossible to convert this into a linear relation for if we impose the condition that the first member shall be a perfect square, we find $m = - 1$, but this reduces the equation to $r + \rho = 0$.

As an illustration of transformation, let us apply this cell to the circle

$$\rho = c.\cos\theta$$

which passes through the pole or fulcrum. Substituting this value of ρ in (3) we have

$$r^2 + (1 - m)\, c\, r \cos\theta - m\, c^2 \cos^2\theta = k^2,$$

or in rectangular coordinates

$$(x^2 + y^2)^2 + (1 - m)(x^2 + y^2)\, c\, x - (m\, c^2 + k^2)x^2 - k^2 y^2 = 0. \; . \; . \; . \; (4)$$

The transformed curve is therefore a bicircular quartic with a node at the origin, the inverse of which with respect to its node is a conic. Hence if (4) be itself transformed by means of a reciprocator attached to the same fulcrum O' the result is a conic.

Now make P' in Fig. 1 the fulcrum and $P'\,O' = \rho$, $P'Q' = r$, then $O'\,Q'$ which is r in equation (3) is denoted by $r - \rho$. Therefore substituting in (3) $r - \rho$ for r we have

$$r^2 - (1 + m)r\rho = m(m + 1)^2(a^2 - b^2) = k^2. \; . \; . \; . \; . \; . \; . \; (5)$$

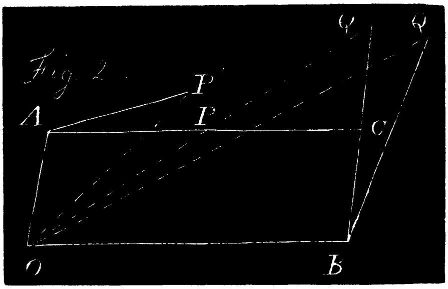
This cell is therefore equivalent to that described on page 43 ANALYST Vol. II. If we take Q' as fulcrum we shall find a similar relation between r and ρ.

A cell similar to Peaucellier's, except that the symmetrically situated pairs of bars are of the three unequal lengths l, m, and n, has been called the *Scalene Cell.* The relation between r and ρ is

$$r\rho(r+\rho)+(m^2-l^2)r+(m^2-n^2)\rho=0. \,.\,.\,.\,.\,.\,.\,.\,. \quad (6)$$

Prof. Cayley has considered the transformation by this cell in an Article in the Quarterly Journal of Mathematics for May 1875; with the results that for a circle passing through the fulcrum, and also for a straight line, the transformed curve is a circular cubic; for the circle in general it is a symmetrical bicircular quartic, but the Cartesian though belonging to the latter class cannot be described in this manner.

A remarkable discovery of Sylvester's is the Plagiograph or Skew Pantagraph, of which he gave an account in Nature for July 1st 1875.



Let $AOBC$ be a jointed parallelogram, and let P and Q be so taken that $AP : PC :: BC : CQ$, then OPQ is a staight line and O being the fulcrum, P and Q are the tracer and follower in the ordinary pantagraph. Now take a point P' rigidly connected with the bar AC, and Q' rigidly connected with BC, and such that $AP'=AP$, $BQ'=BQ$, and the angles $P'AP$, $Q'BQ$ are equal but taken in opposite directions. Then OAP' and OBQ' are similar triangles and the angle $P'OQ'$ (difference between AOB and the sum $AOP' + AP'O$) is equal to PAP' (difference between the supplements AOC and AOP'), that is $P'OQ'$ is constant. Thus the rays OP' and OQ' have a constant ratio and make a constant angle, and if P' describe a given curve, Q' will describe a similar curve, as in the case of the ordinary pantagraph, but *turned or skewed through a given angle.* The rays of the Skew Pantagraph may be made equal by taking P and Q to coincide at C.

This extension of the Pantagraph suggested at once to Prof. Sylvester and to Mr. Kempe an analogous generalization of Mr. Hart's discovery of a four-bar reciprocator. The heavy lines in Figure 3 represent the four bars of Hart's reciprocator, but the points P, P', Q, Q', instead of being taken *on* the bars AD, BC, AB and DC respectively (as in Fig. 5, page 44, ANALYST Vol. II), are taken off but rigidly connected with these bars at the vertices of similar triangles constructed upon them, as represented by the dotted lines in the figure. Now if we join PQ, PQ', $P'Q$, $P'Q'$, it is readily seen that $P'CQ'$ and BCD are similar triangles; in like manner PAQ & DAB are similar, having the same ratio of similitude as the former pair; hence as BCD & DAB are equal; so likewise are $P'CQ'$ and PAQ. Therefore $P'Q' = PQ$. From the triangles PDQ' and $P'BQ$, which can be shown to be equal in the same manner, we have $PQ' = P'Q$. Hence $PQP'Q'$ is a parallelogram. By reason of the similar triangles $P'Q'$ bears a fixed ratio to BD, namely that of $Q'C$ to DC; and PQ' bears a fixed ratio to AC, that of DQ' to DC. Now as proved in the former article the product $BD \times AC = (BC)^2 - (DC)^2$, hence the product $P'Q' \times PQ' =$ *the difference of the squares of the unequal links multiplied by the ratio of the product of the sides of one of the similar triangles to the square of the link on which it is constructed.*

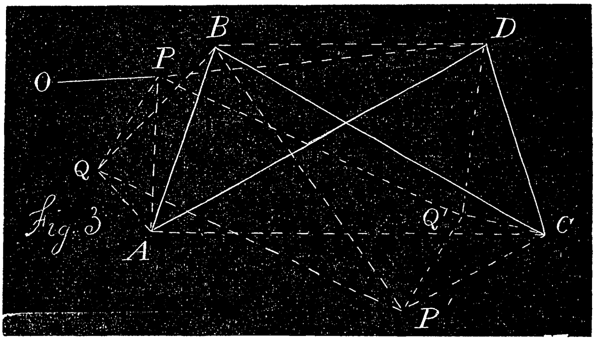


Fig. 3

$PQP'Q'$ is then a parallelogram the product of whose sides is constant; it remains to prove that its angles are constant. The angles $DQ'P$ and $CQ'P'$ being by the similar triangles equal to DCA and CDB, their sum is two right angles; therefore the sum of the remaining angles at Q', namely $PQ'P'$ and $DQ'C$ is two right angles, that is $PQ'P'$ is the supplement of the fixed angle $DQ'C$.

Now taking Q as fulcrum, QP and QP' are rays making a constant angle and inversely proportional to one another, hence if P describe a given curve P' will describe the inverse curve turned through a given angle about the fulcrum. This skew reciprocator has been named by Sylvester the Quadruplane.

If P be constrained by a link OP, pivoted at O so that $OQ = OP$, to move in a circle passing through Q, P' will describe a straight line as in the inversion of the circle by an ordinary reciprocator; but whereas in the ordinary case this line must be perpendicular to the line of centres OQ, by the use of the Quadruplane the line described may make any desired angle with the line of centres.

(*To be continued.*)